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The Comparison of Weighted Residual Methods of Solving Boundary Value Problems in Science and Engineering

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Abstract

In this research article, the comparison of weighted residual methods of boundary value problems are investigated. A second order differential equation within the range $0 \leq x \leq 1$ *is solved for all the methods to verify the efficiency and reliability of the methods. The result shows that the approximate solution which is a linear combination of the basis function (weight function) converges to the real solution depending on whether the basis function is identical, continues, and square-integrable. Also, of the methods investigated, only the Petrov-Galerkin and collocation methods gives a better approximation to the exact solution whose trial function satisfies all essential boundary conditions.*

Keywords: *Weighted Residual, Ordinary Differential Equation, Boundary Value Problems, Residual error, Finite Element method*

Introduction

Several boundary value problems that arise in the field of science and engineering often do not have close form solutions. Such problems are only amenable to approximate solutions which either converges to the exact solution or diverges depending on whether the base function is the same as the trial function. Before the advent of the finite element method widely used by scientists and engineers, there existed an approximation technique used in solving a differential equation called the method of weighted residual (MWR).

The underlying principle behind this methods is to use an approximate solution, which must be analytic or piecewise analytic, minimizes the value of the residual, and is expressible as the linear combination of the base or trial functions chosen from a linearly independent set, whose coefficients are determined by a method of choice. These methods are best suited to solving such a class of problem because they are easy to implement. They include the Galerkin, collocation, least-square, subdomain, and method of moments. The only distinction between these methods is the choice of the weight function.

METHODOLOGY OF THE WEIGHTED RESIDUAL METHODS

Galerkin Method

In this method, the test and trial functions are the same and the weight functions are considered as coefficients of the trial functions

$$w_i(x) = \phi_i(x), i = 0,1, \dots, n$$

$$\langle R, \phi_i \rangle = 0$$

$\int_V \left[L\big(\hat{y}(x)\big) - f(x)\right] \phi_i(x)dx = 0$ or

$$\int_V \phi_i(x)R(x)dx = 0$$

The above results into a system of equations for the unknowns, c_i. Equally, the method is seen as a modified form of the least square method. Instead of taking the derivative of the residual concerning the unknowns, c_i, the weight function is taken as the derivative of the approximate solution, $\tilde{\hat{y}}(x)$ concerning the unknowns

That is, $w_i(x) = \frac{\partial \hat{y}(x)}{\partial c_i} = \phi_i(x)$

Collocation Method

In the collocation method, the weight function is taken from the family of the Dirac delta functions, δ in the domain. That is, $w_i(x) = \delta(x - x_i)$ with the following property

$$\delta(x - x_i) = \begin{cases} 1, & x = x_i \\ 0, & otherwise \end{cases} \qquad i = 0,1, \dots\dots, n$$

Now, the in tergal form multiplication using the above give

$$\int_V w_i(x)R(x)dx = R(x_i)$$

Thus, $R(x_i) = L(\sum_{i=o}^{n} c_i\phi_i(x)) - f(x) = 0$

$$\Rightarrow \sum_{i=0}^{n} c_i L(\phi_i(x)) = f(x_i)$$

$$\Rightarrow R(x_i) = f(x_i)$$

Subdomain Method

In the subdomain method, the domain of interest V is subdivided into nonoverlapping subsections or subdomains of the form, $V_{i+1} = [x_i, x_{i+1}],\ i = 0,1, \dots \dots, n-1$

The weight function is taken as unity in the subdomain and zero elsewhere as follows

$$w_i(x) = \begin{cases} 1, & x \epsilon V \\ 0, & x \notin V \end{cases}$$

This method is considered as a modification of the collocation method. The underlying idea is to force the residual to zero at fixed points as well as various subdomains in the domain to allow for the unknowns to be calculated.

$$\int_V w_i(x)R(x)dx = \sum_i \left(\int_{V_i} R(x)dx \right) = 0$$

Least Square Method

In this method, the equation of residue given above is squared and integrated over the domain of interest as follows.

$$I = \int_V R(x)R(x)dx = \int_V R^2(x, c_i)dx$$

Minimizing the above integral such that the derivatives of I is taken over all the unknown parameters, we have

$$\frac{\partial}{\partial c_i} \int_V R^2(x, c_i)dx = 0$$

Where the $c_i^{\prime s}$ are the unknown coefficients of the approximate solution, $\hat{y}(x) = \sum_{i=0}^{n} c_i\phi_i(x)$

Further reducing the minimized integral gives

$$2\int_V R(x, c_i)\frac{\partial R}{\partial c_i}dx = 0,\ \ i = 0,1, \dots., n$$

$$\Longrightarrow 2\int_V R(x,c_i)\frac{\partial R}{\partial c_i}dx = 0$$

Comparing with the integral from above in the collocation method, the weight function for the least square method is given by

$w_i(x) = \frac{\partial R}{\partial c_i}$ for $i = 0,1,\dots\dots\dots,n$

Method of Moment

In this method, the weight functions are chosen from the family of polynomials and expressed in the form, $w_i(x) = x^{i-1}$, $i = 0,1,2,\dots\dots,n-1$

If the basis function for the approximate solution, $(\phi_i(x) = w_i(x))$ are chosen as polynomial, then the method of moment coincides with the Galerkin method.

Step 1. Assume an approximate solution. If the order of the differential equation is n, then a polynomial of order $(n+1)$ is obtained, and $(n+2)$ undetermined constants

$y(x) \approx \hat{y}(x) = \sum_{i=0}^{n} c_i\phi_i(x)$ (1)

The trial or base functions in Eq. (1) are linearly independent such that $\hat{y}(x)$ satisfies the boundary condition

Alternatively, Eq. (1) may also be written as

$\hat{y}(x) = y_0(x) + \sum_{i=0}^{n} c_i\phi_i(x)$ (2)

Where $y_0(x)$ satisfy the boundary condition and the trial function are chosen such that interpolates the desired solution subject to the boundary condition.

Step 2. Apply boundary condition to the assumed approximate solution, $\hat{y}(x)$ in step 1, and then find the constants, c_i depending on the order of the equation.

Step 3. Substitute the approximate solution, $\hat{y}(x)$ into the given differential equation, $L(\hat{y}(x)) - f(x) \neq 0$. This results in an error

Step 4. The resulted error in step 3. Is called residual given as

$R(x) = L(\hat{y}(x)) - f(x)$ (3)

Step 5. Take the derivative of the approximate solution and substitute into the residual in step 4

Step 6. Choose an arbitrary weight function, $w_i(x)$ which are the coefficient of the approximate solution or residual depending on the chosen method. This is then multiplied by the residual and integrated over the domain of interest as follows

$\int_V w_i(x)[L(\hat{y}(x)) - f(x)]dx = \int_V w_i(x)R(x)dx \neq 0\,,\ i = 01,2,\dots,n$ (4)

The above integral is made as small as possible or vanishes at finite points depending on the weight function.

Step 7. Forcing the integral multiplication form in step 7. To vanish over the entire domain V as follow

$$\int_V w_i(x)R(x)dx = 0 \tag{5}$$

The above results into an $(n+1)$ algebraic equations for the unknowns c_i, where $i = 0,1,\dots,n$

Step 8. Solve the resulting algebraic equation simultaneously for the constants, c_i

Step 9. Put the values of the obtained constants, c_i into the approximate solution, $\hat{y}(x)$ and get an equation in terms of x as the only unknown.

Step 10. Put the values of x given in the equation to obtain the final answer of the approximate solution, $\hat{y}(x)$

ILLUSTRATIVE PROBLEM.

Here we take a question to show the robustness of the weighted residual methods to the exact solution

Example 2.1 Solve $\frac{dy}{dx} = x\,, 0 \le x \le 1$ subject to the condition, $y(0) = 1, y(1) = 1$

Solution. **Galerkin Method**

Given $\frac{d^2y}{dx^2} = 3x - 4y,\ y(0) = 0, y(1) = 1$ (6)

$\Longrightarrow \frac{d^2y}{dx^2} = 3x - 4y = 0$

Let the approximate solution of the given equation be

$$\hat{y} = \hat{y} = c_0 + c_1x + c_2x^2 + c_3x^3 \tag{7}$$

Putting the boundary condition $y(0) = 0$ and $y(1) = 1$ gives $c_0 = 0, c_1 = 1 - c_2 - c_3$

The approximate solution is now of the form

$$\hat{y} = x + (x^2 - x)c_2x + (x^3 - x)c_3 \tag{8}$$

Differentiating Eq. (8) $w.r.t.x$, twice and substituting into eq. (6) we get the residual error as

$$R = (4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x \neq 0 \tag{9}$$

Hence the inner product of the weighted function and residual error becomes, $\langle w_{i,}, R\rangle = 0$

$\int_0^1 w_i R dx = 0$ (10)

For $i = 1,\ w_1 = x^2 - x$

$\int_0^1 (x^2 - x)[(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$ reduced to

$12c_2 + 18c_3 = -5$ (11)

Similarly, for $i = 2, w_2 = x^3 - x$ we obtain

$$\int_0^1 (x^3 - x)[(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$$

$-\frac{3}{10}c_2 - \frac{52}{105}c_3 = \frac{2}{15}$ (12)

Solving Eqs. (11)−(12), we get the constants

$c_2 = -0.141224$ and $c_3 = 0.183628$

The approximate solution is given as

$\hat{y} = 0.9576x - 0.141224x^2 + 0.183628x^3$

Least-Square Method

Using the differential with respect to the constants, $w_i = \frac{\partial R}{\partial c_i}$, we have the weighted function as

$$w_1 = \frac{\partial R}{\partial c_1} = 4x^2 - 4x + 2$$

$$w_2 = \frac{\partial R}{\partial c_2} = 4x^3 + 2x$$

Substitution of w_1 and w_2 into the integral form in Eq. (10) we have

$\int_0^1 (4x^2 - 4x + 2)[(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$ yields

$28c_2 + 42c_3 = -10$ (13)

Similarly, for $i = 2, w_2 = 4x^3 + 2x$

$$\int_0^1 (4x^3 + 2x)[(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$$

$294c_2 + 716c_3 = -154$ (14)

Solving Eqs. (13) − (14) yields

$c_2 = -0.08987$ and $c_3 = -0.1782$

Substituting the above into the approximate solution gives

$\hat{y} = 1.26807x - 0.08987x^2 - 0.1782x^3$

Subdomain Method

Subdividing the domain $(0,1)$ into two nonoverlapping subintervals $\left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$ and then integrating over the residual error of the form gives

$$\int_0^{1/2} [(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$$

$$\frac{2}{3}c_2 + \frac{5}{16}c_3 = -\frac{1}{8} \quad (15)$$

$$\int_{1/2}^{1} [(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$$

$$\frac{2}{3}c_2 + \frac{27}{16}c_3 = -\frac{3}{8} \quad (16)$$

Solving Eqs. $(15) - (16)$ gives the constants as $c_2 = -0.102299$ and $c_3 = -0.181752$

Approximate solution becomes

$\hat{y} = 1.28405x - 0.102299x^2 - 0.181752x^3$

Moment Method

Putting the weighted functions $w_1 = 1$ and $w_2 = x$ into the integral form multiplication gives

$\int_0^1 [(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$ yields

$$\frac{4}{3}c_2 + 2c_3 = -\frac{1}{2} \quad (17)$$

$\int_0^1 (x)[(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$ yields

$$\frac{2}{3}c_2 + \frac{22}{15}c_3 = -\frac{1}{3} \quad (18)$$

Solving Eqs. $(17) - (18)$, the constants become

$c_2 = -0.1071$ and $c_3 == 0.1786$

The approximate solution now becomes

$\hat{y} = 1.2857x - 0.1071x^2 - 0.1786x^3$

Petro-Galerkin Method

For $i = 1, w_1 = x$

$\int_0^1 (x)[(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$ yields

$$10c_2 + 22c_3 = -5 \quad (19)$$

For $i = 2, w_2 = x^2$, the integral for multiplication now become

$$\int_0^1 (x^2)[(4x^2 - 4x + 2)c_2 + (4x^3 + 2x)c_3 + x]dx = 0$$

$$\frac{7}{15}c_2 + \frac{7}{6}c_3 = -\frac{1}{4} \quad (20)$$

Solving Eqs. $(19) - (20)$ gives $c_2 = -0.2381$ and $c_3 = -0.1190$

The approximate solution of the equation now become

$\hat{y} = 1.3571x - 0.2381x^2 - 0.1190x^3$

Point Collocation Method

Here we subdivide the domain of interest $(0,1)$ into two collocation points and force the residual to zero at these points

Hence, $R\left(\frac{1}{3}\right) = 0$. This yields the algebraic equation

$$30c_2 + 22c_3 = -9 \quad (21)$$

Similarly, $R\left(\frac{2}{3}\right) = 0$

$$30c_2 + 68c_3 = -18 \quad (22)$$

Solving Eqs. (21) and (22), we obtain the constants $c_2 = -0.1565$ and $c_3 = -0.1957$

Substitution of the above into the approximate solution gives

$\hat{y} = 1.3522x - 0.1565x^2 - 0.1957x^3$

Therefore, the approximate solutions of the weighted residual methods are

Galerkin method: $\hat{y} = 0.9576x - 0.141224x^2 + 0.183628x^3$

Least Square method: $\hat{y} = 1.26807x - 0.08987x^2 - 0.1782x^3$

Subdomain method: $\hat{y} = 1.28405x - 0.102299x^2 - 0.181752x^3$

Moment method: $\hat{y} = 1.2857x - 0.1071x^2 - 0.1786x^3$

Petro-Galerkin method: $\hat{y} = 1.3571x - 0.2381x^2 - 0.1190x^3$

Collocation method: $\hat{y} = 1.3522x - 0.1565x^2 - 0.1957x^3$

RESULTS AND DISCUSSION

The comparison of weighted residual methods for the solution of boundary value problems in engineering and science taking into consideration a second order differential equation has been studied analytically. In this section, we discuss the variations between the exact solution and the approximate solution from the methods. The errors produced between the respective methods and exact solution is presented in Tables 1-6 and Figures $(1) - (6)$

Table 3.1 Comparison of Galerkin Method solution and Exact Solutions

x	Exact Solution, $y(x)$	Galerkin Method	Error= $\|y(x) - \hat{y}(x)\|$
0	0.00000	0.00000	0.00000
0.1	0.129622	0.094531	0.0350907
0.2	0.257066	0.187339	0.0697265
0.3	0.380241	0.279527	0.100715
0.4	0.497228	0.372195	0.125033
0.5	0.606352	0.466446	0.139906
0.6	0.706253	0.563381	0.142872
0.7	0.795937	0.664102	0.131835
0.8	0.87482	0.769711	0.105109
0.9	0.942747	0.88131	0.0614375
1.0	1.00000	1.00000	0.000000

Table 3.2 Comparison of Least Square Method solution and Exact Solution

x	Exact Solution, $y(x)$	Least Square Method	Error= $\|y(x) - \hat{y}(x)\|$
0	0.00000	0.0000	0.00000
0.1	0.129622	0.12573	0.00389156
0.2	0.257066	0.248594	0.00847212
0.3	0.380241	0.36752	0.127201
0.4	0.497228	0.481444	0.0157841
0.5	0.606352	0.589293	0.0170595
0.6	0.706253	0.689998	0.0162549
0.7	0.795937	0.78249	0.134470
0.8	0.87482	0.865701	0.00911951
0.9	0.942747	0.938561	0.00418677
1.0	1.00000	1.00000	1.1102XE-16

Table 3.3 Comparison of Subdomain Method and Exact Solutions

x	Exact Solution, $y(x)$	Subdomain Method	Error= $\|y(x) - \hat{y}(x)\|$
0	0.00000	0.00000	0.00000
0.1	0.129622	0.12720	0.0024214
0.2	0.257066	0.251264	0.0058017
0.3	0.380241	0.371101	0.00914063
0.4	0.497228	0.48562	0.0116081
0.5	0.606352	0.593731	0.0126207
0.6	0.706253	0.694344	0.0119086
0.7	0.795937	0.786368	0.00956957
0.8	0.87482	0.868712	0.00610869

0.9	0.942747	0.940286	0.00246167
1.0	1.00000	0.999999	0.000001

Table 3.4 Comparison of Moment Method solution and Exact Solution

x	**Exact Solution, $y(x)$**	**Moment Method**	**Error= $\|y(x) - \hat{y}(x)\|$**
0	0.00000	0.00000	0.00000
0.1	0.129622	0.12732	0.00230126
0.2	0.257066	0.251427	0.00563852
0.3	0.380241	0.371249	0.00899261
0.4	0.497228	0.485714	0.115145
0.5	0.606352	0.593750	0.012602
0.6	0.706253	0.694286	0.0119661
0.7	0.795937	0.786251	0.00968593
0.8	0.87482	0.868573	0.00624751
0.9	0.942747	0.94018	0.00256767
1.0	1.00000	1.00000	0.00000

Table 3.5 Comparison of Petro-Galerkin Method solution and Exact Solution

x	**Exact Solution, $y(x)$**	**Petro-Galerkin Method**	**Error= $\|y(x) - \hat{y}(x)\|$**
0	0.00000	0.00000	0.00000
0.1	0.129622	0.13321	-0.00358834
0.2	0.257066	0.260944	-0.0038728
0.3	0.380241	0.382488	-0.00224659
0.4	0.497228	0.497128	0.000100121
0.5	0.606352	0.60415	0.00220196
0.6	0.706253	0.70284	0.00341254
0.7	0.795937	0.792484	0.00345313
0.8	0.87482	0.872368	0.00245231
0.9	0.942747	0.941778	0.000969274
1.0	1.00000	1.0000	0.00000

Table 3.6 Comparison of Collocation Method solution and Exact Solution

x	**Exact Solution, $y(x)$**	**Collocation Method**	**Error= $\|y(x) - \hat{y}(x)\|$**
0	0.00000	0.00000	0.00000
0.1	0.129622	0.133459	-0.00383764
0.2	0.257066	0.262614	-0.00554868
0.3	0.380241	0.386291	-0.0604969
0.4	0.497228	0.503315	-0.00608708

0.5	0.606352	0.612513	-0.00616054
0.6	0.706253	0.712709	-0.00645626
0.7	0.795937	0.80273	-0.00679277
0.8	0.87482	0.881402	-0.00658129
0.9	0.942747	0.94755	-0.00480243
1.0	1.00000	1.00000	0.000000

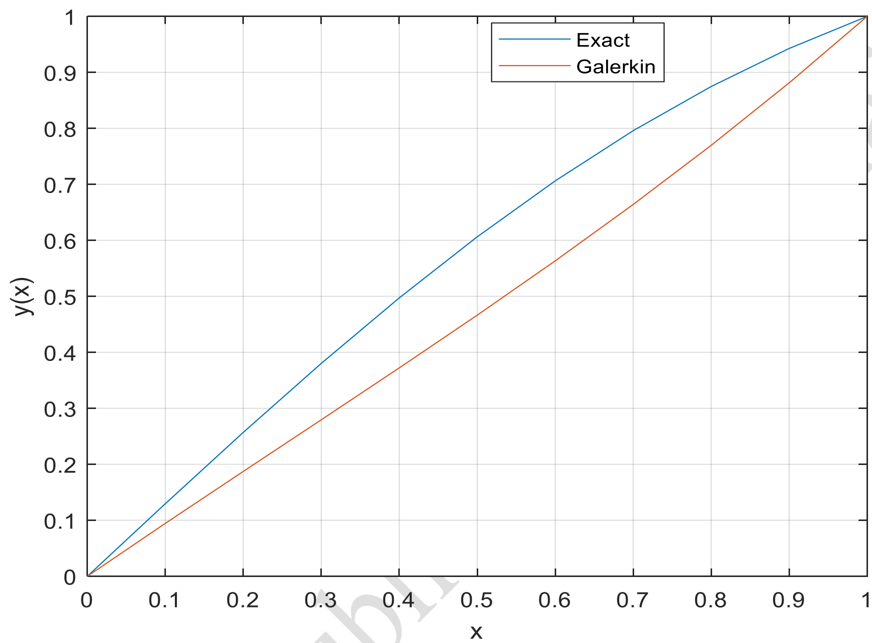

Figure 1. Comparison of Exact Solution and the Galerkin Solution of $y^{11} + 4y - 3x = 0, y(0) = 0, y(0) = 1$

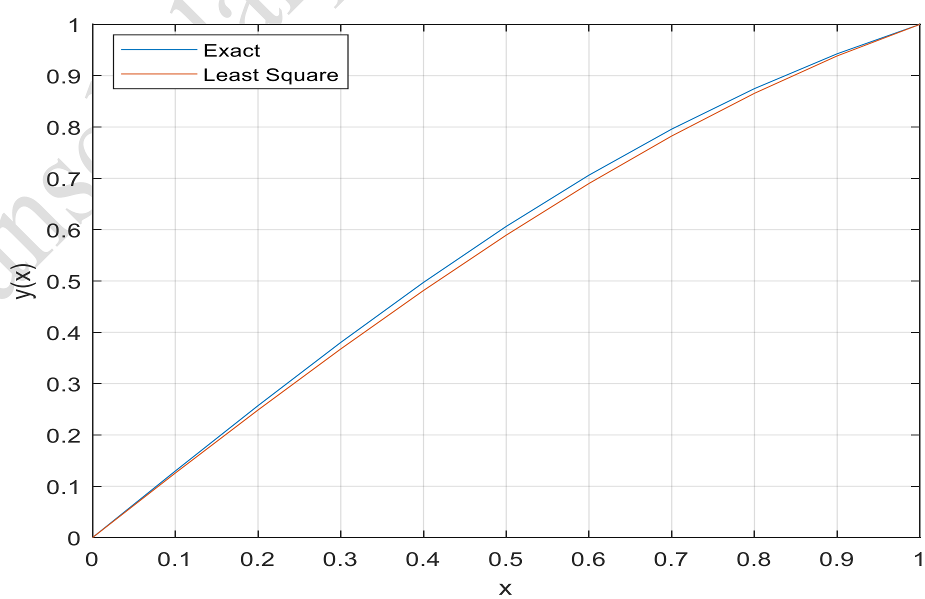

Figure 2. Comparison of Exact Solution and Least Square solution of $y^{11} + 4y - 3x = 0, y(0) = 0, y(0) = 1$

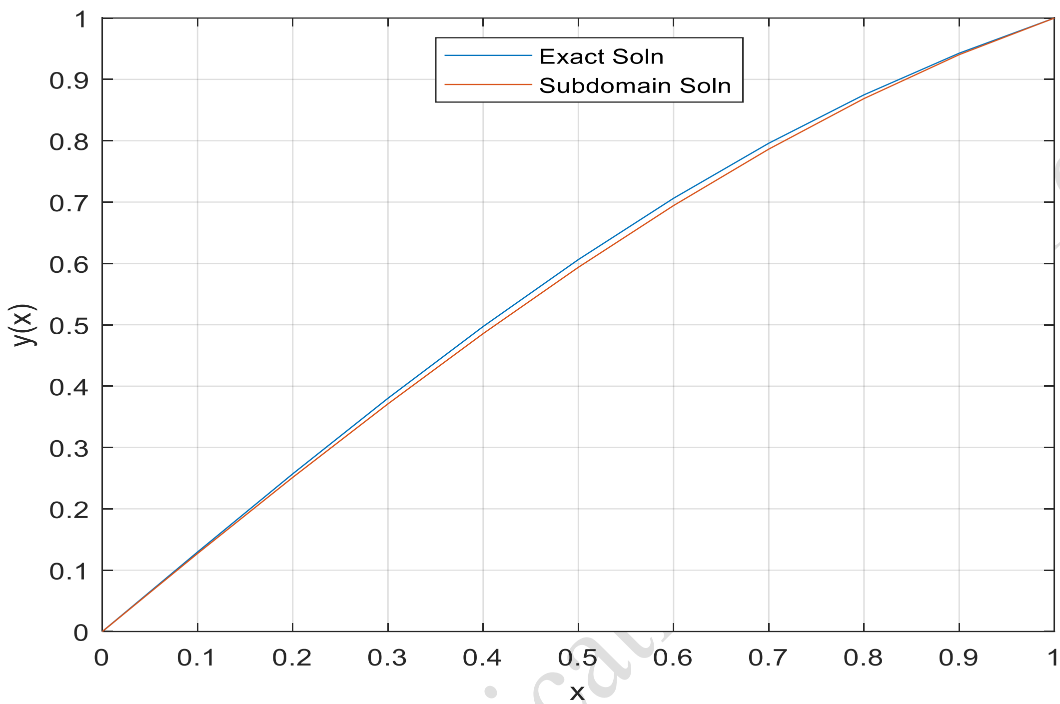

Figure 3. Comparison of Exact Solution and Subdomain Method of $y^{11} + 4y - 3x = 0, y(0) = 0, y(0) = 1$

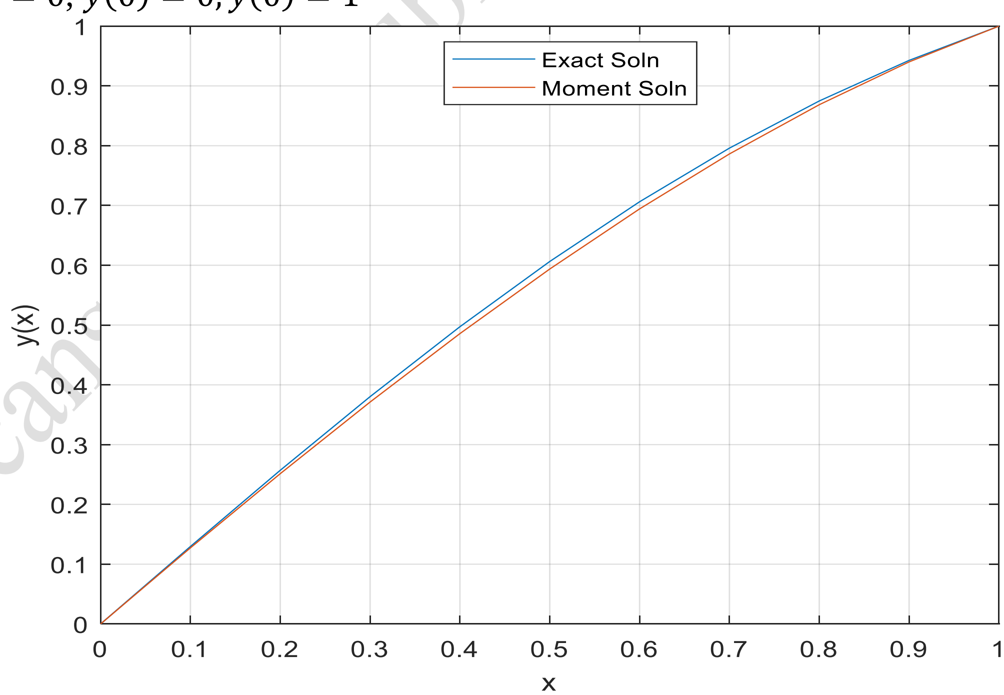

Figure 4. Comparison of Exact Solution and Moment Method of $y^{11} + 4y - 3x = 0$, $y(0) = 0, y(0) = 1$

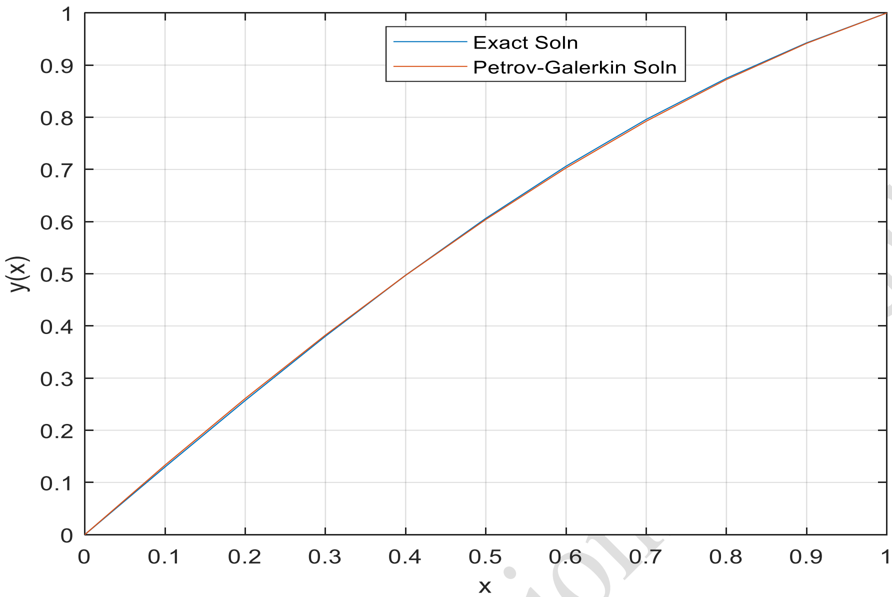

Figure 5. Comparison of Exact Solution and Petrov-Galerkin Method of $y^{11} + 4y - 3x = 0$, $y(0) = 0, y(0) = 1$

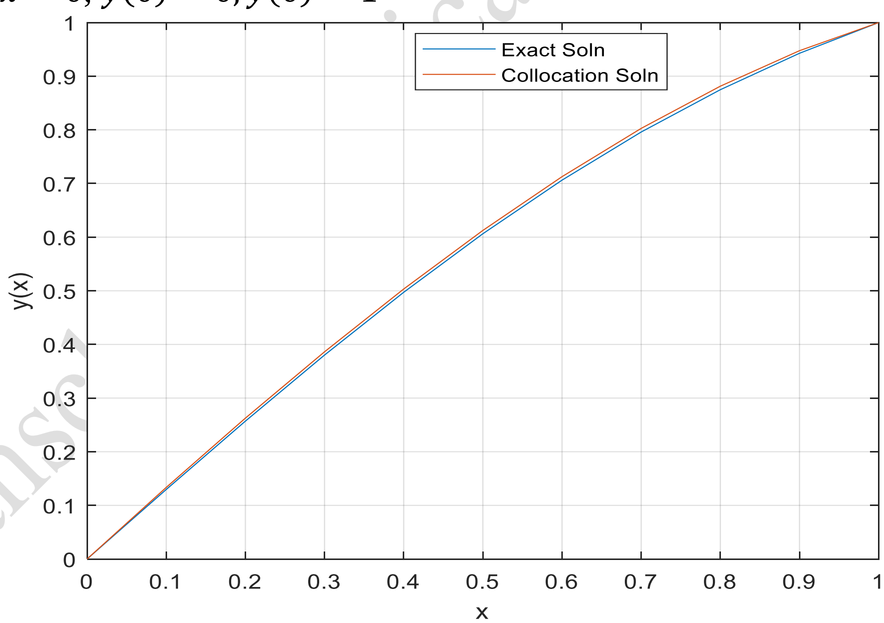

Figure 6. Comparison of Exact Solution and Collocation Method of $y^{11} + 4y - 3x = 0$, $y(0) = 0, y(0) = 1$

CONCLUSION

The comparative study of the methods of weighted residual is investigated theoretically and the results are presented graphically. The result shows the Petrov-Galerkin, Collocation, moment and subdomain methods gives solution with an acceptable accuracy, whereas, the least square and Galerkin methods produces solution with bigger error, which explains why they are widely preferred and used than the other weighted residual methods. Overall, they all produce a computable error and easily assemble the linear system. The results obtained agree with results in literature. We recommend higher order boundary value problems be examined by further researchers to verify if the result obtained is consistent or the marked error between the approximate and exact solutions for the Galerkin and least square methods is affected by the order of the equation.

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